Chords, scales, and divisor lattices

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Abstract

In this paper I study musical harmony from the rational point of view, stressing the significance of integer ratios. The main concept is the *divisor set* of an integer. A divisor set is the set of all divisors of a number and it has the algebraic structure of a lattice with greatest common divisor and least common multiple as the lattice operations. A divisor set can also be interpreted as a geometric lattice, a set of points with integer coordinates in the space of prime exponent sequences (‘tonal space’), with the shape of a parallelepiped (‘shoebox’, ‘brick’). Divisor lattices can also be seen as set intersections of harmonic (‘overtone’) series with subharmonic (‘undertone’) sequences.

I shall concentrate in particular on the divisor lattices of three numbers, 60, 8640, and 345600 with 12, 56, and 120 divisors, respectively. In the tonal space they are bricks with side lengths 2×1×1, 6×3×1, and 9×3×2, respectively. Projecting these point sets on the ‘pitch line’ gives the following musically recognizable sets: major and minor triad chords and the tonic seventh chord, the diatonic scale, and the just chromatic scale, respectively. The equation of the pitch line in parametric form is $\mathbf{p} \log \mathbf{p_i}$ where $\mathbf{p_i}$ is the $\mathtt{i^{th}}$ prime and $\mathbf{p}$ is the pitch parameter.

The tonal space

We define the **tonal space** as the set of rational numbers $\mathbb{Q}$ represented as the exponent vectors (sequences) of primes. The fundamental theorem of arithmetic allows us to write every rational $f \in \mathbb{Q}$ uniquely as

$$f = \prod_{i=1}^{\infty} p_i^{e_i}$$

where p_i is the i^{th} prime and e_i is the multiplicity of p_i in the prime factorization of f. The logarithmic pitch corresponding to the frequency f is then

$$\log f = \sum_{i=1}^{\infty} e_i \log p_i$$

and this form suggests that we interpret the pitch as an inner product $\boldsymbol{e} \cdot \boldsymbol{p}$ between a *tonal vector* $\boldsymbol{e}$ and a constant *pitch vector* $\boldsymbol{p}$ whose i^{th} component is the logarithm of the i^{th} prime. Slightly more abstractly, the tonal space can be defined as the infinite-dimensional real vector space $\mathbb{R}^{\infty}$ with a distinguished non-zero vector $\boldsymbol{p}$. One can then choose a basis in which this vector has the logarithms of primes as components. The integer lattice points $\mathbb{Z}^{\infty}$ in this basis are called *tonal points*.

The advantage of this representation is that whereas divisibility relations are lost when taking ordinary logaritms, prime-wise logarithms preserve them. In particular, for two tonal points $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{a}$ divides $\boldsymbol{b}$, $\boldsymbol{a} \mid \boldsymbol{b} \Leftrightarrow a_i \leq b_i$ for all i.

The tonal space geometrizes the sets of rational intervals, chords and scales, and aids in visualizing complex harmonic structures. The infinitude of primes does not bother us because the traditional Western music theory makes use of the first three prime dimensions 2, 3, and 5 only. It will be interesting anyhow to extend the study to the next higher primes 7, 11, and 13 and study whether we can hear the 'eerie' harmonic qualities of the higher dimensions. If not, it is an interesting question why just three primes suffice. Does the reason lie in the neural structures of our auditory system, or is it just a historical accident?

The natural 'prime basis' of the tonal space consists of the orthogonal unit vectors $\mathbf{e}_2$, $\mathbf{e}_3$, and $\mathbf{e}_5$ corresponding to the prime numbers 2, 3, 5, … It is useful to introduce another set of basis vectors $\mathbf{e}_p$, $\mathbf{e}_u$, $\mathbf{e}_v$ called the 'pitch basis,' where the first unit vector is parallel to the pitch vector and the remaining ones orthonormal with it. In the 3D case there is arbitrariness in choosing the direction of the second basis vector but once that choice is made, the third basis vector is determined by orthogonality.

The following graphics shows the prime basis, the pitch vector **p**, and one pitch plane orthogonal to **p**.

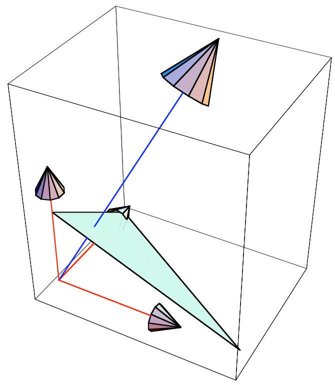


Fig. 1. Prime basis, pitch vector, and pitch plane. The 2-axis is to lower right, the 5-axis to upper left.

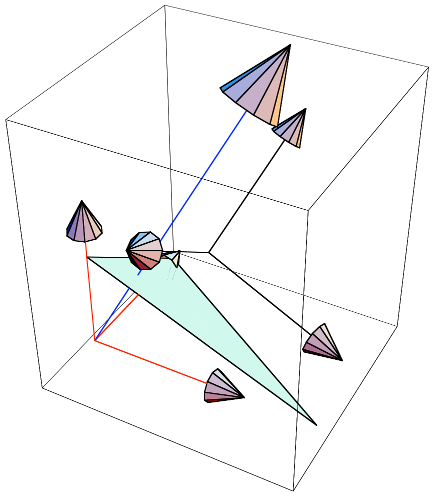
The first new base vector $\mathbf{e}_p$ is obtained by dividing the pitch vector by its length: $\mathbf{e}_p = \mathbf{p} / |\mathbf{p}|$

The second unit vector $\mathbf{e}_u$ is chosen from the plane othogonal to **p**; in the direction of the projection of $\mathbf{e}_2$. The third pitch basis vector $\mathbf{e}_v$ is obtained as the vector product of the first two.

We construct the tonal rotation matrix **P** with the pitch basis vectors as rows.

$$\mathbf{P} = \begin{pmatrix} \mathbf{e}_p \\ \mathbf{e}_u \\ \mathbf{e}_v \end{pmatrix} = \begin{pmatrix} 0.335136 & 0.531178 & 0.778161 \\ 0.94217 & -0.188943 & -0.276797 \\ 0 & -0.825924 & 0.563781 \end{pmatrix}$$

The symbolic expressions for the matrix entries are not terribly complicated but here we give only numerical values. Any tonal space vector given in prime coordinates can be represented in the pitch basis by multiplying it with this matrix. The first component will be the pitch, the second u-component might be called the 'octavicity' and the third v-component gives the mixture between'dominantness' and 'mediantness.' The lone zero in the rotation matrix reflects our particular choice of the u-axis. One can easily calculate the pure dominantness and pure mediantness by first projecting $\mathbf{e}_3$ and $\mathbf{e}_5$ to the enharmonic pitch plane, normalize them, and use inner products with these projections.



Fig, 2. Prime basis, pitch vector, pitch plane, and pitch basis. Actually, the origins of the two coordinate systems coincide. Here, for clarity, the pitch basis is drawn shifted to an arbitrary location.

Divisor lattices

The divisor set `Divisors[n]` of a natural number will play a central part in what follows. In mathematics, the term *lattice* has two distinct meanings, an algebraic and a geometric one. In our case of tonal space geometric representation of the algebraic divisor lattice of a natural number, these two notions are unified. In the following, we give graphical images of divisor lattices in the 'pitch-up' orientation, transformed by **P** to pitch coordinates. We connect the points representing two numbers m and n by a line if one of them **divides immediately** the other, i.e. one is a prime times the other.

Here is an example of a divisor lattice, that of the number 60. This structure will be analyzed in more detail in the next section.

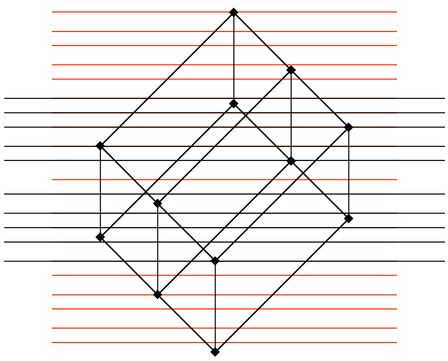


Fig. 3. The triad divisor lattice of 60 with fundamental $\mathtt{C}_1$ and leading tone $\mathtt{B}_6$. Vertical dimension is strictly logarithmic, thence the diatonic staff and ledger lines are unequally spaced.

All positive integers make up an infinite divisor lattice $\mathbb{N}^\infty$. It could be called the 'overtone lattice.' The next diagram is the bottom corner of this infinite (and infinite-dimensional) lattice showing the first four dimensions up to 60 (but omitting numbers that are not 7-smooth, i.e., contain a prime divisor larger than 7).

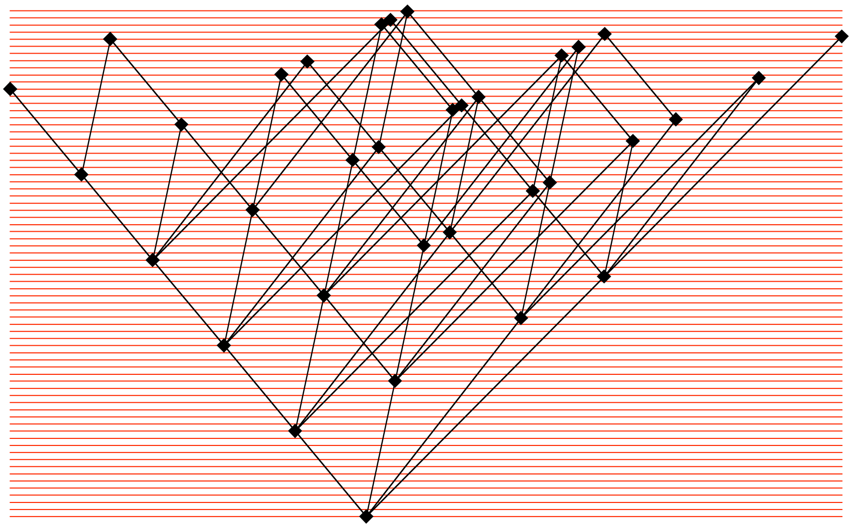


Fig. 4. "overtone series" is an infinite divisor lattice. Here is a lattice diagram of it the bottom corner of it, up to the number 60. The number 49 situated at top right

Here are all divisor lattices up to 100 in musical notation, of 5-smooth numbers (that is, numbers not containing a prime divisor higher than 5.)

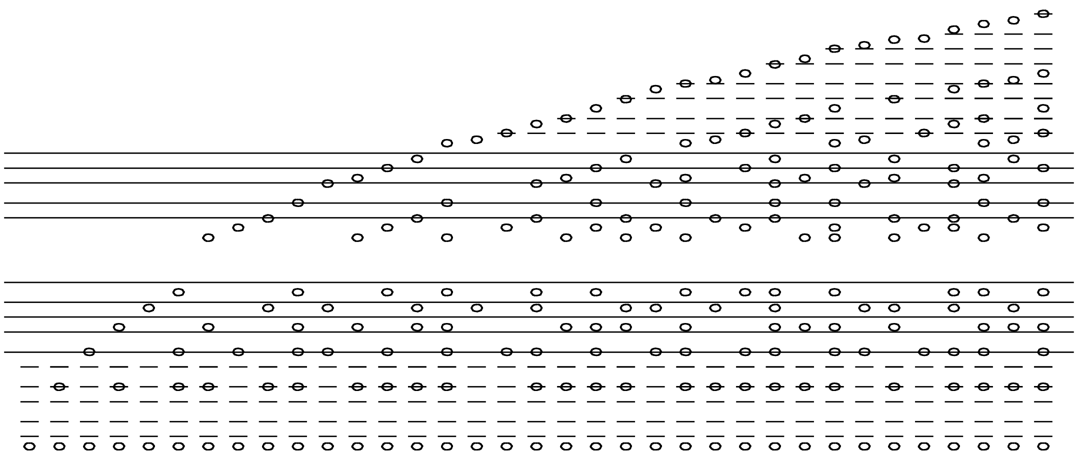


Fig. 5. Divisor lattices of all 5-smooth numbers up to 100, written starting from $\mathtt{C_1}$.

The triad lattice

Major and minor triads in their root positions both appear in the divisor lattice of $60 = 2^2\ 3^1\ 5^1$. This number appears as the number base of ancient Babylonians and still today in time and angle measurements, probably because it has a large number of divisors. This is due to a good mixture of small prime factors. Observe that powers of 2 have the smallest prime divisors but all of these being the same, they do not yield particularly many *different* divisors. They only give octave scales.

The major triad 4.5:6 and the minor triad 10:12:15 appear both symmetrically in the middle of the divisor set of 60: {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60}.

In music notation this combined major/minor chord (with C_1 as the fundamental) appears as follows.

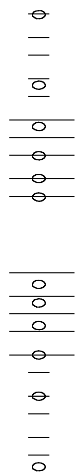


Fig. 6. The triad lattice of divisors of 60.

A note on notation. This system of notation does not use any accidentals. The vertical scale is pure logarithmic, not diatonic (except for a gap just under the middle C. "for lyrics"). Observe that the spaces between staff lines are unequal. In the narrow space there is room for two pitch levels, in the large for three. By using right sized note heads, these cases are easy to distinguish by looking if a space note touches the line above it, below it, or neither.

One can see here also the C seventh chord, in a particular setting, with the third and the fifth doubled and the C-major leading tone well separated from the root. In this case it is natural to consider the C_1 as the fundamental (the greatest common divisor, gcd, of chord frequencies) and equally natural although less obvious, the B_6 (the least common multiple, lcm) as the actual *leading tone*. Because of two prime factors 2 in 60, the fundamental is octave doubled twice and, correspondigly, the leading tone (and similarly every other tone).

The difference in sensory qualities of major and minor can be explained nicely if one makes the following (unproven) hypothesis about the mechanism of hearing. *When we hear a tone with reasonably many dominant harmonic spectral components, we compute their common fundamental (gcd) and their common leading tone (lcm) and then 'prime' the hearing apparatus to pitches in the divisor set, as if a set of tuning forks in the brain were tuned to all frequencies that are multiples of the fundamental and submultiples of the leading tone*.

This computation is well-known in the context of the 'missing fundamental' problem [see the research by the Dutch pioneers, Plomp & Levelt]. Whether the same phenomenon is known in the context of a 'missing leading tone,' I am not aware.

Under this hypothesis, it is easy to explain why major is 'bright and light' and minor 'dark and subdued.' In fact the hypothesis predicts that the C major chord and the e minor chord as positioned above, are equally bright. But if one compares the C major chord and a c minor chord from the same root, the minor chord activates a set of internal resonators to an identical divisor chord which is tuned lower by a twelfth.

The two standard inversions of the triads do not fit in the divisor set of 60, they both require the larger harmony 120 and are in this sense less consonant. Surprisingly, the divisor theory predicts that there are more consonant settings of the triads, than the root positions. Consider the divisor set of 30, {1, 2, 3, 5, 6, 10, 15, 30}

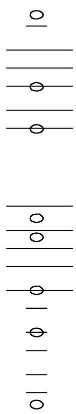


Fig. 7. The "mellow triads" in the divisor lattice of 30.

Here one finds the major triad `C2, G2, E3` as a major sixth above a fifth, and the minor triad as a fifth above a major sixth. I call these chords the "mellow triads." In the mellow major triad, the third is transposed up by an octave, in the mellow minor triad the third is transposed down by an octave. The reader is asked to aurally test whether my claim of mellowness is true or not. Comparing the standard and mellow triads on a piano is not very conclusive because of the rich spectrum of a piano tone. Use sines instead.

The number 30 is actually the smallest 3-dimensional number because it contains all three smallest primes 2, 3, and 5, each just once. In the tonal space its divisor lattice is represented by the unit cube. The larger elongated parallelepiped of 60 is simply the 'octave dounbling' of the 'mellow harmony' 30.

The nearest tonal extensions of the lattice 60 are obtained by enlarging the 2×1×1 box of 60 by one unit in each tonal dimension, in turn. Notated versions of the lattices 60, 120 = 2 × 60, 180 = 3 × 60, and 300 = 5 × 60 are as follows.

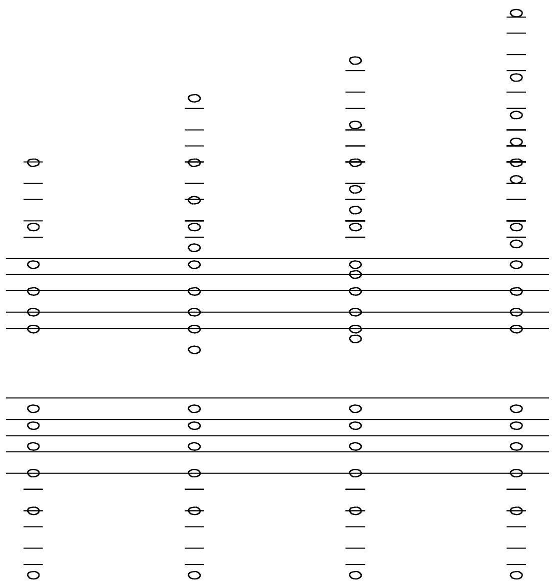


Fig. 8. Divisor lattices of 120, 180, and 300 give "enriched" triad harmonies.

Notice that in 120 the seventh chord (C-E-G-B) appears in the middle symmetrically. Being itself symmetrical (major third, minor third, major third), this is the only place in a divisor set where it can exist without being mirrored somewhere else. We conclude that this chord itself is major-minor neutral. We generalize and claim that the attributes 'major' and 'minor' actually refer to the lower half and the upper half of a divisor lattice, respectively. From the other two extensions 180 and 300 the reader may want to read a few interesting things about what the lattice theory predicts next in terms of 'graciously sounding' chords in a given tonality.

It may also be instructive to look at the minimal extensions of the mellow harmony 30. We get the lattices, 60, 90, and 150.

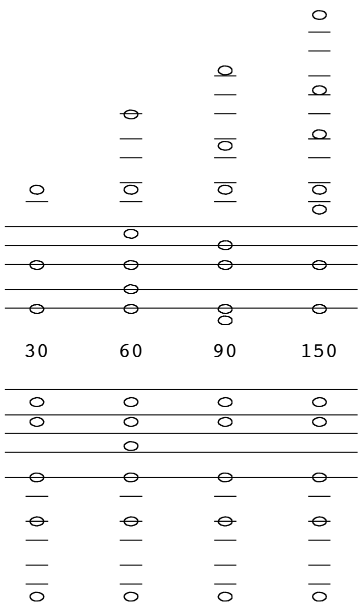


Fig. 9. Enriched lattices of the mellow triad lattice 30.

The diatonic lattice

We turn to the study of the divisor set of one particular number, $8640 = 2^6\ 3^3\ 5^1$. It has 56 divisors. On a linear scale the divisors grow fast as a function of their index.

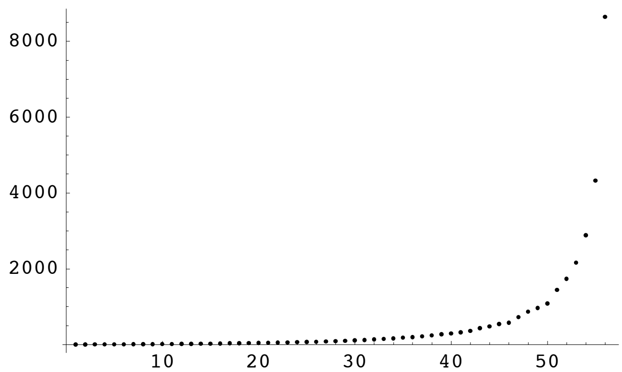


Fig. 10. Divisors of the diatonic number 8640.

A logarithmic plot reveals a sigmoid shape typical of divisor sets. The central almost linear segment gives something that resembles the diatonic musical scale (Fig. 11). The vertical scale is given in octaves. The full divisor lattice of 8640 exceeds the human auditory range by three octaves.

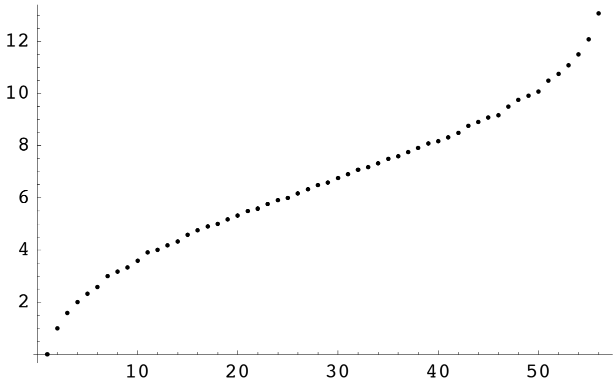


Fig. 11. Logarithmic plot of of the lattice 8640.

A logarithmic plot of the intervals between consecutive divisors shows the central linear segment as the flat bottom of a bathtub shaped curve. The vertical scale units are now equally tempered semitones.

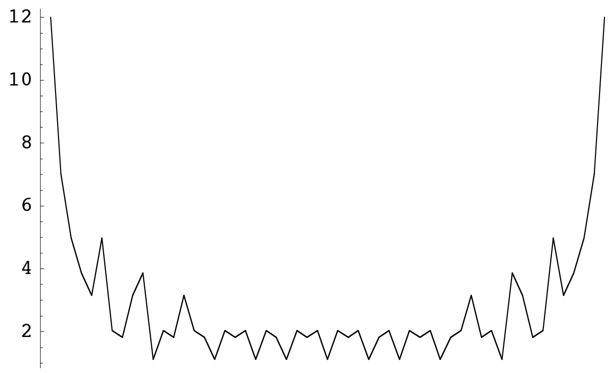


Fig. 12. Intervals between consequtive divisors of 8640, in semitones. The central set of "teeth" reproduce the tone/semitone structure of the diatonic scale.

The central tooth pattern reflects the alteration of the diatonic scale intervals 16/15, 10/9, and 9/8.

The ordinary just major scale is obtained as the 15^{th} through 22^{nd} divisors

{24, 27, 30, 32, 36, 40, 45, 48}

The standard normalized form is obtained by dividing these nubers by the first number 24.

$$\left\{1, \frac{9}{8}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{15}{8}, 2\right\}$$

In analogy with the lattice 60, we expect to obtain the just minor diatonic scale as the inversion mirror of the major scale. We get

{180, 192, 216, 240, 270, 288, 320, 360}

$$\left\{1, \frac{16}{15}, \frac{6}{5}, \frac{4}{3}, \frac{3}{2}, \frac{8}{5}, \frac{16}{9}, 2\right\}$$

The mild surprise here is that we obtain the natural minor scale with its *second degree flattened,* that is, the *phrygian mode*. (This was pointed out to me by Rolf Wallin.)

The following graphic (major scale left, minor scale right) shows how the principal three degrees (I, IV, V) are common to both scales. All other pitches are shifted downwards from major to minor, by roughly equal amounts, and as a result, fill in the octave into 12 very roughly equal steps, except for a central gap for the missing tritonus..

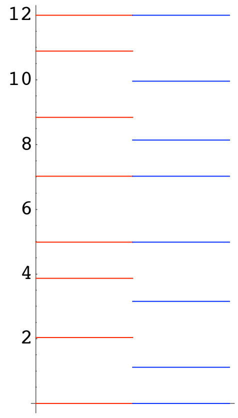


Fig. 13. The major and minor scales from 8640 compared.
They are inversion mirrors of each other.

It is instructive to look at the interval structures of the tonal neighbors of 8640, obtained by dividing and multiplying it by the primes 2, 3, and 5. These tonal relatives are 1728, 2880, 4320, 17280, 25920, and 43200. We shall not do that analysis here.

The full lattice of 8640 is reproduced in Fig. 14.

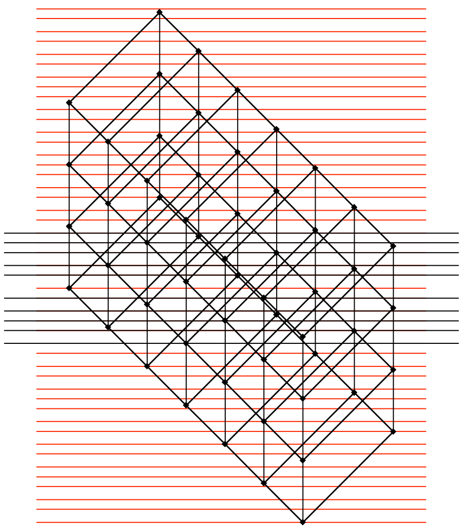


Fig. 14. Divisor lattice of 8640 gives the diatonic scale. The three ‘floors’ othis ‘building’ correspond to the tonal functions IV, I, and V (counted from bottom up).

The just chromatic scale

What is the just chromatic scale? What are the exact rational pitches of the black keys? Textbooks are generally silent about this and published values of scale degrees are not consistent. The standard answer to this dilemma is that the actual chromatic pitches depend on the tonal environment, but this is not a satisfying answer. Why is it not possible to specify the tonal environment?

I did a systematic computer search for a number among whose divisors there would be a long (> 12) sequence of intervals of one semitone (when the intervals were rounded to the nearest ET semitone), and found the smallest such number to be 345600 = $2^9\ 3^3\ 5^2$. I call this the “Donald Duck number” because it looks ridiculous but is not hard to remember.

The ambitus of this divisor set is ~18.4 octaves. If the set is centered around the middle C, the fundamental is at ~ 0.4 Hz and the leading tone at ~ 170 kHz.

Taking the lowest sequence of its divisors with semitone intervals, the 52^{nd} to 64^{th} of its 120 divisors, we obtain the following *just chromatic scale*:

{360, 384, 400, 432, 450, 480, 512, 540, 576, 600, 640, 675, 720}.

The first number is 360, again familiar from angular measurements. The normalized scale degrees are:

$$\left\{1, \frac{16}{15}, \frac{10}{9}, \frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{64}{45}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \frac{16}{9}, \frac{15}{8}, 2\right\}$$

All intervals are familiar, except for the strange looking 64/45 for the tritonus, a nice rational approximation to $\sqrt{2}$. The scale intervals are

$$\left\{\frac{16}{15}, \frac{25}{24}, \frac{27}{25}, \frac{25}{24}, \frac{16}{15}, \frac{16}{15}, \frac{135}{128}, \frac{16}{15}, \frac{25}{24}, \frac{16}{15}, \frac{135}{128}, \frac{16}{15}\right\}$$

We have four different semitones here, with sizes measured in ET semitones

{0.707, 0.922, 1.117, 1.33}

One of them, 16/15, is the *diatonic semitone*. The smallest one, 25/24 is the *chromatic semitone*. The remaining two are 27/25, the *Pythagorean great limma*, and 135/128, the *major chroma*.

We took the divisors 52^{nd} to 64^{th} from the **bottom half** of the divisor set. For this reason the scale obtained is the *major just chromatic scale*. Had we taken the scale from the mirror symmetrical upper part of the divisor set, we would have obtained the *minor just chromatic scale:*

{480, 512, 540, 576, 600, 640, 675, 720, 768, 800, 864, 900, 960}

The following graphic gives an indication of how much the just major scale and the just minor scale differ. The gray lines mark ET semitones.

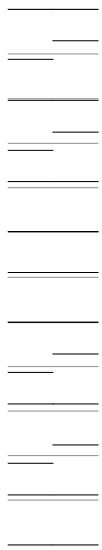


Fig. 15. Major (left) and minor (right) just chromatic scales compared. The gray lines give the equally tempered (ET) semitone values.

Note: This graphic should be remade for better clarity, there may also be an error in the choice of the minor scale.

We had chosen the 52nd divisor as the lowest pitch of the major just chromatic scale because in the full divisor set it marks the first divisor in the sequence of 18 divisors with semitone distances. (There are 17 consecutive ones in the following list.) But if you carefully look at the pattern, you may agree that the "tight JCS" begins considerably earlier, there are just a few gaps in it, marked by 2's in the list. The just scale gradually dissolves into a diatonic scale at its both ends.

```
{12, 7, 5, 4, 3, 5, 2, 2, 3, 4, 1, 2, 2, 3, 1, 1, 2, 1, 2, 2, 2, 1, 1, 1, 2, 1, 2, 1, 1, 2, 1,
 1, 1, 2, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 2, 1, 1, 1, 2,
 1, 1, 2, 1, 2, 1, 1, 1, 2, 2, 2, 1, 2, 1, 1, 3, 2, 2, 1, 4, 3, 2, 2, 5, 3, 4, 5, 7, 12}
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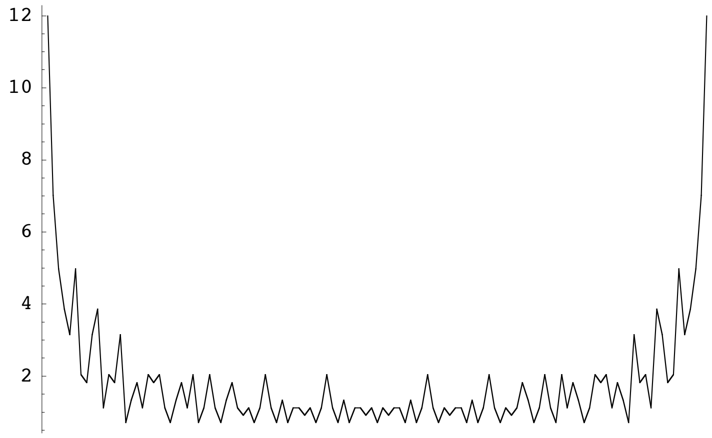
The idea that one begins with the lowest continuous stretch of semitones is artificial. Just increasing the size of the lattice will give a different result. The following two curves indicate this. The first is the ‘bathtub curve’ for 345600, the second for 4 times it (1382400).



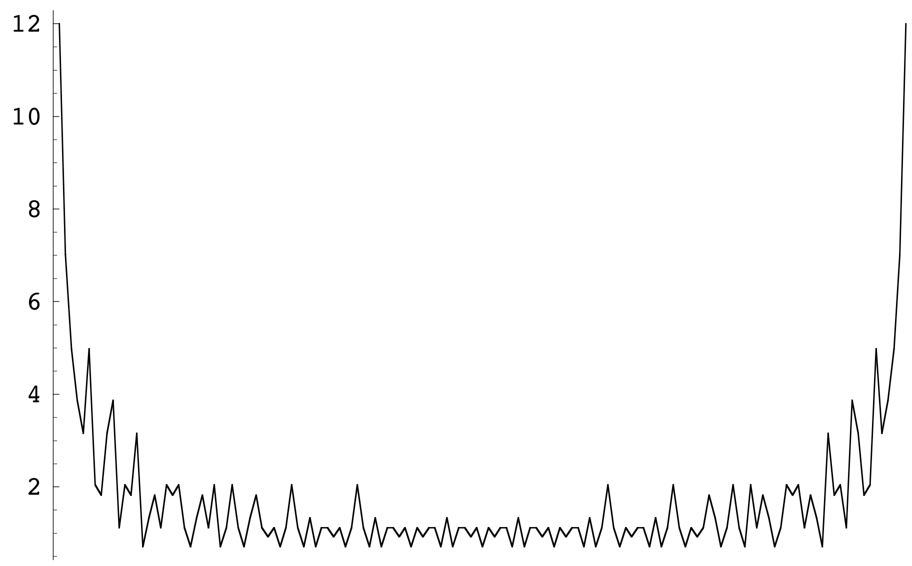


Fig. 16. Bathtub curves for 345600 and 1382400.

By octave enlarging the set, the upward teeth going to 2 semitones get more separated and we obtain a longer central stretch of chromatic scale. The interval structures of the flanking diatonic stretches look interesting and may deserve separate study.

We have, in several cases, identified *majorness* with the lower half of a divisor lattice, starting from the bottom **fundamental**, and *minorness* with the upper half of a divisor lattice, ending to the top **leading tone**. (In earlier writings I called the highest number the "sampler," because it would be a natural sampling frequency for all pitches in its lattice. I have lost the reference to the musicologist stating that "Tonality is a thing determined by a fundamental pitch and a leading tone." That gave me the kick to identify the highest pitch (least common multiple) of a divisor lattice with the leading tone.

Let us introduce a more exact measure of majorness / minorness, the M-index ('majmin index'). The function applies to any rational chord or scale; it gives a number in the range -1 … +1 (-1 for extreme major, +1 for extreme minor, 0 for majmin neutral). It is constructed as follows. Let c denote a rational chord. Compute its gcd(c) and lcm(c). Construct a straight line scale σ(p) on the logarithmic pitch p such that σ(log gcd(c)) = -1 and σ(log lcm(c)) = +1. Then compute the weighted sum M(c) = $\sum_{i \in c} p_i \, \sigma(p_i)$. Results for typical chords are given in the following table.

Chord	Relative frequencies	Major/minor index
major triad root position	4 5 6	-0.220471
major triad first inversion	5 6 8	-0.236811
major triad second inversion	6 8 10	-0.333333
minor triad	10 12 15	0.220471
mellow major triad	2 3 5	-0.333333
6 first partials	1 2 3 4 5 6	-0.464363
major seventh	$1 \ \frac{5}{4} \ \frac{3}{2} \ \frac{15}{8}$	0
minor seventh	$1 \ \frac{6}{5} \ \frac{3}{2} \ \frac{15}{8}$	0.126357
dominant seventh	$1 \ \frac{5}{4} \ \frac{3}{2} \ \frac{16}{9}$	-0.142885

Table 1. The major / minor indices for some chords.

It may seem odd that the ordinary triads have such low value of majorness/minorness, less than one fourth of the maximum. The mellow triad fares as well as the second inversion. But if one takes 1000 first partials (harmonic overtones), the index will be -0.988. Thus, the extremal values ±1 are approached asymptotically by long overtone and undetone series. The diatonic major and minor scales have absolute values 0.155 and the just chromatic scales 0.013 . All 2-chords (intervals) have a zero index, of course, like all inversion symmetric chords. There are no 'major' or 'minor' intervals despite of verbal usage. Actually, it should be an interesting task to clarify the reasons for classifying intervals into major and minor ones.

Higher tonal dimensions

Why is the tonal space only 3-dimensional? I have not seen any well-researched case of a musical system in use that would have its practice (instruments and music) and theory, that essentially were using primes higher than 5. Powers of 7 and higher primes may occur in theoretical calculations. The Chinese 'changing note' *pien* may reflect occasionally the seventh partial, but it may also reflect the usual confusion about flattening the seventh degree. This confusion still exists in the confusion about what is the minor scale. An interesting hypothesis is that the jazz 'blue note' would be a natural seventh.

Musical instruments are still constructed not to sound prime partials 7, 11, 13 and higher. We want to know why music is based on 5-smooth numbers.

Note. A natural number is *k-smooth* if it does not contain a prime divisor larger than k.

Here are some possible explanations to the 3-dimensionalty of the musical tonal space, all still quite hypothetical.

1. Brain 3-dimensionality. One hypothesis is that somewhere in the ascending auditory pathway, e.g. the cochlear nucleus, the olivary nucleus, or the inner geniculate nucleus, there exists a 3-dimensional frequency division network, with the evolutionary function to bundle audio spectrum partials to groups corresponding to separately moving sound sources. The 3-dimensionality of the neural network precludes the independent use of more than 3 tonal dimensions.

2. 3 dimensions generate sufficiently dense scales. Contra-argument: the 2-dimensional Pythagorean space $2^{\mathbb{Z}}\ 3^{\mathbb{Z}}$ is already dense). For some reason people consider the Pythagorean tuning to be 'odd.'

3. Computational tradeoff in brain processing.

4. Plain tradition. Instrumentalists and singers might well use natural sevenths, but nobody has registered this usage.

5. Plain tradition. The natural septimal seventh might be acceptable but unfortunately it is not available from standardized instruments and the small intervals between it and established scale degrees would introduce unpleasant dissonance.

6. More than 3 dimensions would overload the auditory processing system.

7. Hearing might use a modified cortical structure, originally evolved for spatial tasks. Sound objects originally moved in 3-space and their auditory counterparts might have from the beginning gained advantage by utilizing existent genetic machinery for 3-space representations.

Musically, the question is whether tonal dimensions 2, 3, and 5 really are separate dimensions of musical qualia, and whether 7, 11, and so on would represent 'higher' and new suach qualia. Consider the first question first. The difference between the divisor set and tonal space ideas and traditional theoretical views is that I have taken the octave dimensional on equal footing with the 3-dimension ('fifth-dimension', dominant-subdominant dimension) and 5-dimension ('third-dimension', mediant dimension). I do not have experimentally valid facts about the comparability of the 'octavicity', 'dominantness', and 'mediantness' dimensions, just my own perceptions. Mathematically, it just seems that the traditional music theory has unnecessarily mixed the octave dimension with the 3- and 5-dimensions by taking the fifth and the third intervals as the generators, instead of the plain primes.

By calling the space of prime exponent the ‘tonal space,’ I have implicitly made the assumption that the prime exponents were really representing different qualia, in a best possible case, ‘independent’ or ‘orthogonal qualia.’ Depending on how our brains are constructed, the fourth dimension of 7 might sound, if well used musically, as plain dissonant mess, as indifferent, or as a new kind of ethereal tonal quality. This can be tested, but the test paradigm should take into account that our musical hearing apparatuses might well be biased towards a 3-dimensional interpretation because of lifelong exposure.

The following graphic [Fig. 17] indicates how badly prime partials fit a 12-tone scale. The ET degrees are the longer cyan lines, the partials short black lines. The seventh partial is nearer to the major seventh than to the minor seventh, likewise the 14^{th}. The next two primes 11 and 13 sit crudely in almost quarter-tone positions.

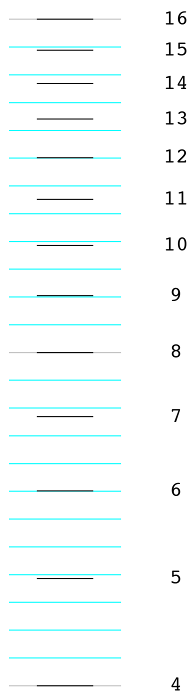


Fig. 17. Partials 4 - 16 and ET pitches.

Tonal shapes

The present discussion has centered on particular simple shapes in the tonal space of prime exponent sequences, the divisor lattices. Their advantage is mathematical simplicity, but perhaps neural circuits in the brain do not especially appreciate right angles. There are several plausible ways to generalize the divisor lattice concept towards more general ‘potato shapes’ (triaxial ellipsoids, rotated triaxial ellipsoids, spherical harmonics, convex sets, etc.). For still more generality, define a ‘tonal distribution’ as a complex-valued function on the tonal space, $\rho: \mathbb{Z}^\infty \longrightarrow \mathbb{C}$. It assigns with every tonal vector $\mathbf{x}$ a sine wave with amplitude $|\rho(\mathbf{x})|$ and phase $\mathrm{Arg}(\rho(\mathbf{x}))$. The general claim is now that the concepts of tonality and tonal center are captured by the notion of a tonal distribution which is convex (in a suitable sense) and has a single maximum.

Hypothesis. This is actually the strongest assumption in this article. Evolution has found a way that empowers us to deduce from pressure variations at two head-mounted points (meatuses) an internal description of a sound space, weakly mapped on the visual-kinesthetic space, of sound sources and reflectors. I have proposed that distinct sound objects correspond to smooth humps (maxima) in tonal space distributions. The auditory apparatus groups all sound partials nearby to each well-defined activation maximum into a single group with a perceived tonal center, each such group potentially corresponding to a separate individual sound source in the environment.

The local tonal environment

Look at the tonal point environment of the origin, in the pitch coordinate system {p, u, v}. The lattice points are not any more neatly arranged, after the rotation to this new coordinate system. But their geometric configuration is the same for every tonal point, this is a universal structure. Let us try to visualize the tonal environment of a lattice point. We have here a mild conflict between the two notions of 'what is the distance of two pitches?', the nearness in pitch or the enharmonic tonal nearness. We try to form a distance function that balances between the two measures in such a way that two pitches are judged to be near to each other if 1) their tonal components are similar, or, 2) their pithches are nearby. This approach will lead to the idea of a local tonal environment in the shape of a rotation hyperboloid.

Begin by considering cylinder shaped environments in the pitch coordinate system. For simplicity, consider a cylinder with height *h* in the pitch dimension and a circular shape with radius *r* in the enharmonic subspace. Here are two graphics showing the lattice points in a flat 'pancake' cylinder and in an oblong 'sausage-like' cylinder:

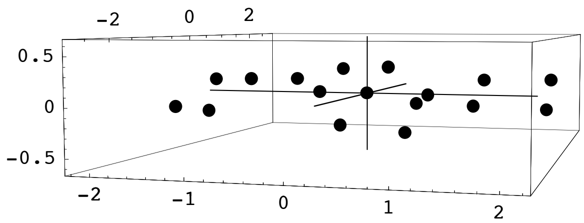


Fig. 18. Flat "pancake" tonal environment = melodic neighbors.

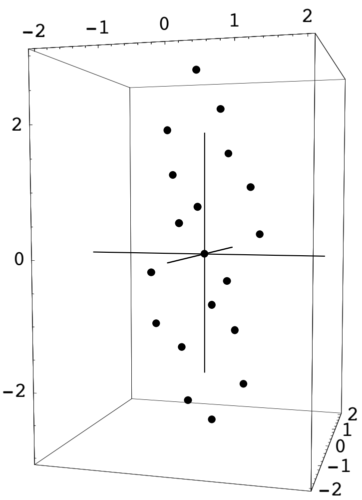


Fig. 19. Oblong "sausage" tonal environment = a scale.

Although the two examples of Figs. 18 and 19 were chosen rather randomly, exactly the same patterns exist around every point in the tonal space.

Coincidences

It seems that all the curious tonal structures in music result from number-theoretic coincidences. A coincindence is simply a rational number very near to unity. For example the long-time (theoretical) success of the Pythagorean tuning system derives from the fact that 2^{19} is very nearly equal to 3^{12}, their ratio being the small Pythagorean comma 1.01364 semitones which is small enough to justify the false statement that the circle of fifthes closes in 12 steps. The syntonic comma 81/80 similarly derives from the nearness of strong and weak whole tones 9/8 and 10/9.

I did a systematic computer search on coincidences in low tonal dimensions. Here are some examples.

For example, in Fig. 20. there is a plot of a (suitably weighted) measure of the sameness of powers of 3 and 5. The exponent of 3 varies from 1 to 50 and that of 5 from 1 to 35. The prominent peak near to the origin reflects the sameness of 25 and 27, or the second power of 5 and third power of 3. There seems to be a general repetitive pattern of peaks.

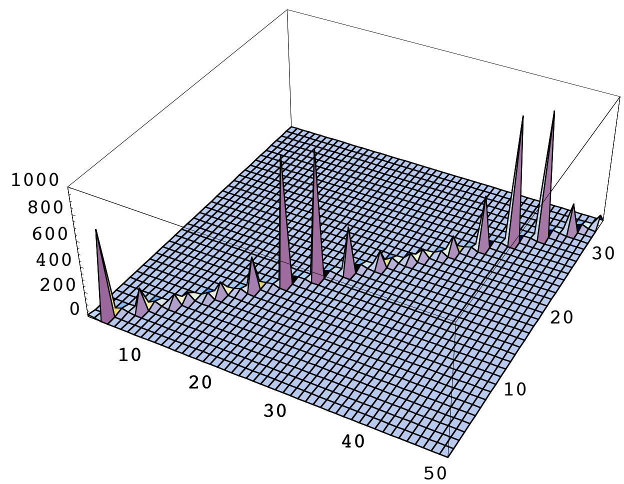


Fig. 20. Coincidence map of powers of 3 and 5. A peak in the plot means that some power of 3 (right down axis) is nearby to a power of 5 (right up axis).

The following table gives some results from a 4-dimensional search. The leftmost column gives the tonal coordinates, the exponents of primes 2, 3, 5, 7. The middle column gives the rational number interval, and the right column the interval in ET semitones.

{-9, 6, 1, -1}	$\frac{3645}{3584}$	0.292178
{-9, 8, -4, 2}	$\frac{321489}{320000}$	0.0803696
{-6, -10, 7, 2}	$\frac{3828125}{3779136}$	0.222978
{-5, -3, 3, 1}	$\frac{875}{864}$	0.21902
{-5, 2, 2, -1}	$\frac{225}{224}$	0.0771152
{-4, 9, -2, -2}	$\frac{19683}{19600}$	0.0731577
{-1, -9, 9, -2}	$\frac{1953125}{1928934}$	0.215766
{-1, -7, 4, 1}	$\frac{4375}{4374}$	0.00395756
{0, -5, 1, 2}	$\frac{245}{243}$	0.141905
{1, 2, -3, 1}	$\frac{126}{125}$	0.137948
{5, -4, 3, -2}	$\frac{4000}{3969}$	0.134693
{6, -2, 0, -1}	$\frac{64}{63}$	0.272641
{6, 0, -5, 2}	$\frac{3136}{3125}$	0.0608324
{7, 5, -4, -2}	$\frac{31104}{30625}$	0.268683
{7, 7, -9, 1}	$\frac{1959552}{1953125}$	0.0568749
{10, -6, 1, -1}	$\frac{5120}{5103}$	0.057578

Table 2. Near approximations to unity in the 4-dimensional tonal space.

There is one 'Pythagorean' beauty $\frac{64}{63}$ and a few other pleasant-looking ones. Look at that $\frac{4375}{4374}$, coincidence to 4 thousandths of a semitone!

The mathematical tonal space extends to infinity in all tonal dimensions, but we have limited ability to distinguish nearby pitches. This also depends on the tempo of the music. Short notes do not possess a definite pitch, either mathematically of cognitively. The result is that the tonal volume of distinguishable pitches becomes finite, a compactified space. A systematic study of the actual shape and size of the evoked tonal distribution, as a function of tempo and note durations, in time, will be a major future undertaking.

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